LESSON 18 - STUDY GUIDE

ABSTRACT. In this lesson we will focus on the subclass of functions for which the corresponding Fourier series is absolutely convergent. This is one of the strongest, and almost ideal, forms of convergence, as it corresponds to Lebesgue integrability on the integers side of the Fourier coefficients, yielding uniform and $L^p(\mathbb{T})$ convergence in norm, for all $1 \leq p \leq \infty$. We will see that the subset of continuous functions for which absolute convergence of the Fourier series occurs, denoted by $\mathbb{A}(\mathbb{T})$, is an algebra, known as the Wiener algebra, and we will try to characterize these functions in terms of Hölder-α regularity.

1. **Absolutely convergent Fourier series: the Wiener algebra.**

Study material: We will follow very closely the section **6 - Absolutely Convergent Fourier Series** from chapter **I - Fourier Series on** $\mathbb{T}$, corresponding to pgs. 31–33 in the second edition [1] and pgs. 33–35 in the third edition [2] of Katznelson's book.

From Proposition 1.3 in Lesson 15 we know that every trigonometric series that converges absolutely is a Fourier series, for its coefficients necessarily are the Fourier coefficients of the continuous function to which it converges uniformly. So the class of absolutely convergent Fourier series coincides with absolutely convergent trigonometric series, and is therefore general, in this respect.

We will then denote by $\mathbb{A}(\mathbb{T}) \subset C(\mathbb{T})$ the subset of continuous functions whose Fourier series converge absolutely,

$$f \in \mathbb{A}(\mathbb{T}) \Leftrightarrow \sum_{-\infty}^{\infty} |\hat{f}(n)| < \infty.$$

Therefore, the Fourier transform becomes a bijective map

$$\mathcal{F} : \mathbb{A}(\mathbb{T}) \to l^1(\mathbb{Z}),$$

with

$$\mathcal{F}(f)(n) = \hat{f}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) e^{-int} dt,$$

and inverse

$$\mathcal{F}^{-1}(\hat{f})(t) = f(t) = \sum_{-\infty}^{\infty} \hat{f}(n) e^{int},$$

which we can use to pullback to $\mathbb{A}(\mathbb{T})$ the $l^1(\mathbb{Z})$ norm, turning the bijection into a Banach space isometry,

$$\|f\|_{\mathbb{A}(\mathbb{T})} = \sum_{-\infty}^{\infty} |\hat{f}(n)| = \|\hat{f}\|_{l^1(\mathbb{Z})}.$$

Of course we have

$$\|f\|_{L^1(\mathbb{T})} \leq \|f\|_{L^\infty(\mathbb{T})} = \sup_{t \in \mathbb{T}} \left| \sum_{-\infty}^{\infty} \hat{f}(n) e^{int} \right| \leq \sum_{-\infty}^{\infty} |\hat{f}(n)| = \|f\|_{\mathbb{A}(\mathbb{T})},$$

so that the $\mathbb{A}(\mathbb{T})$ norm is even stronger than the supremum norm of continuous functions.

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From an abstract harmonic analysis point of view we have sort of reversed roles. It is as if we are now shifting the focus to $(\mathbb{Z},+)$ as the main locally compact abelian group under consideration, with the counting measure as its Haar measure, and considering the absolutely convergent Fourier series as a Fourier transform defined by an $l^1(\mathbb{Z})$ integral over the integers (with the unimportant sign change in the exponential),

$$f(t)=\int_{n\in\mathbb{Z}}\hat{f}(n)e^{int}.$$

So, from this point of view, $\mathbb{A}(\mathbb{T})$ is just the range of this Fourier transform in the reverse direction, for Lebesgue integrable functions on $\mathbb{Z}$, i.e. sequences in $l^1(\mathbb{Z})$. In other words, $A(\mathbb{T})$ is analogous to $\mathcal{F}(L^1(\mathbb{T}))$, but considering now the series as the Fourier transform from functions on integers $\mathbb{Z}$ to functions on the circle $\mathbb{T}$.

Our first result concerns products of functions in $\mathbb{A}(\mathbb{T})$. We know that the Fourier transform maps convolutions to products, so the reversal of roles should now imply that convolutions on the integers side get mapped by the inverse Fourier transform to products on the circle side. And, as convolutions are indeed well defined products for Lebesgue integrable functions, their image should therefore remain in $\mathbb{A}(\mathbb{T})$.

Proposition 1.1. *Let $f,g\in\mathbb{A}(\mathbb{T})$. Then $fg\in\mathbb{A}(\mathbb{T})$ and*

$$\|fg\|_{\mathbb{A}(\mathbb{T})}\leq\|f\|_{\mathbb{A}(\mathbb{T})}\|g\|_{\mathbb{A}(\mathbb{T})}.$$

Proof. Through the bijection $\mathcal{F}:\mathbb{A}(\mathbb{T})\to l^1(\mathbb{Z})$ the function $f\in\mathbb{A}(\mathbb{T})$ corresponds to $\hat{f}\in l^1(\mathbb{Z})$ and $g\in\mathbb{A}(\mathbb{T})$ corresponds to $\hat{g}\in l^1(\mathbb{Z})$. The convolution $\hat{f}*\hat{g}$ is therefore well defined in $l^1(\mathbb{Z})$ (of course we proved this in Lesson 10 for convolutions of functions in $L^1(\mathbb{R})$, but the proof for $l^1(\mathbb{Z})$, with integrals replaced by absolutely convergent series is exactly the same) and from the L^p estimates for convolutions we have

$$\|\hat{f}*\hat{g}\|_{l^1(\mathbb{Z})}\leq\|\hat{f}\|_{l^1(\mathbb{Z})}\|\hat{g}\|_{l^1(\mathbb{Z})}.$$

So, to prove the result, we just need to confirm that the inverse Fourier transform of $\hat{f}*\hat{g}$ is fg. But it is again totally analogous to proving that the Fourier transform of the convolution equals the product of the Fourier transforms,

$$\begin{aligned}\mathcal{F}^{-1}(\hat{f}*\hat{g})(t) &= \sum_{n=-\infty}^{\infty}\left(\sum_{j=-\infty}^{\infty}\hat{f}(n-j)\hat{g}(j)\right)e^{int}=\sum_{j=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty}\hat{f}(n-j)e^{int}\right)\hat{g}(j)\\ &= \sum_{j=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty}\hat{f}(n)e^{int}\right)\hat{g}(j)e^{ijt}=f(t)\sum_{j=-\infty}^{\infty}\hat{g}(j)e^{ijt}=f(t)g(t).\end{aligned}$$

Of course, we could as easily have proved that $\widehat{fg}(n)=(\hat{f}*\hat{g})(n)$, which amounts to the same. So this concludes the proof. □

So, we conclude that $\mathbb{A}(\mathbb{T})$ is an algebra. It is called the *Wiener algebra*. Obviously, it is just the isometric image of the Banach algebra formed by $l^1(\mathbb{Z})$ with the convolution product, by the injective inverse Fourier transform. More generally, one calls Wiener algebra the image of L^1 with the convolution product by any Fourier transform. So, the image of $L^1(\mathbb{T})$ by the Fourier transform would also be a Wiener algebra of sequences on $\mathbb{Z}$, a subspace of $l^\infty(\mathbb{Z})$. And more commonly, in $\mathbb{R}^n$ one also calls the image $\mathcal{F}(L^1(\mathbb{R}^n))\subset C(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ the Wiener algebra associated to the Fourier transform there.

A difficult problem is how to characterize the elements of the Wiener algebra, in $C(\mathbb{T})$. We already know, from the decay properties of the Fourier transform, that if f is smooth enough, then its Fourier coefficients will decrease sufficiently fast to zero as $|n|\to\infty$ in order for the Fourier series to be absolutely

convergent. So, for example, if $f \in C^2(\mathbb{T})$ then $\hat{f}(n) = o(1/n^2)$ and surely the corresponding Fourier series will converge absolutely, so that $f \in \mathbb{A}(\mathbb{T})$. But a simple exercise using the $L^2(\mathbb{T})$ properties of the derivative yields a better result.

Proposition 1.2. *Let f be absolutely continuous, so that f' exists a.e. $t \in \mathbb{T}$. Let also $f' \in L^2(\mathbb{T})$. Then, $f \in \mathbb{A}(\mathbb{T})$ and*

$$\|f\|_{\mathbb{A}(\mathbb{T})} \leq \|f\|_{L^1(\mathbb{T})} + \left(2\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{\frac{1}{2}} \|f'\|_{L^2(\mathbb{T})}.$$

So, in particular, we have $C^1(\mathbb{T}) \subset \mathbb{A}(\mathbb{T}) \subset C(\mathbb{T})$. But, surprisingly, this result can even be improved down to Hölder-$\frac{1}{2}$ regularity, in spite of the slow decay $1/n^{\frac{1}{2}}$ that such functions have, as we saw in last lesson, which would naturally lead one to believe it to be impossible.

Theorem 1.3. (Bernstein's Theorem) *Let $f \in C^{0,\alpha}(\mathbb{T})$ be a Hölder-α continuous function, for some $\alpha > \frac{1}{2}$. Then $f \in \mathbb{A}(\mathbb{T})$ and*

$$\|f\|_{\mathbb{A}(\mathbb{T})} \leq C_\alpha \|f\|_{C^{0,\alpha}(\mathbb{T})}, \tag{1.1}$$

where the constant C_α depends only on α, and $\|f\|_{C^{0,\alpha}(\mathbb{T})}$ is the Hölder-α norm of f, given by

$$\|f\|_{C^{0,\alpha}(\mathbb{T})} = \sup_{t\in\mathbb{T}} |f(t)| + \sup_{t,s\in\mathbb{T}} \frac{|f(t)-f(s)|}{|t-s|^\alpha},$$

corresponding to the optimal constant in the Hölder-α continuity definition.

Proof. The Fourier series associated to the difference $f(t-h) - f(t)$ is

$$f(t-h) - f(t) \sim \sum_{n=-\infty}^{\infty} (e^{-inh} - 1)\hat{f}(n)e^{int}.$$

Of course at this point we still do not know whether this series converges absolutely or not, so this is really just a statement about the Fourier coefficients, although we already know that it converges in the $L^2(\mathbb{T})$ norm. If we now group the frequencies in dyadic blocks[1] $2^m \leq |n| < 2^{m+1}$ and consider $h = \frac{2\pi}{3}\frac{1}{2^m}$ then $\frac{2\pi}{3} \leq |nh| < \frac{4\pi}{3}$ and therefore

$$|e^{-inh} - 1| = |e^{-i\frac{2\pi}{3}\frac{n}{2^m}} - 1| \geq \sqrt{3},$$

so that

$$\begin{aligned}
\sum_{2^m\leq|n|<2^{m+1}} |\hat{f}(n)|^2 &\leq \sum_{2^m\leq|n|<2^{m+1}} |e^{-inh}-1|^2|\hat{f}(n)|^2 = \|f(\cdot-h)-f(\cdot)\|^2_{L^2(\mathbb{T})} \\
&\leq \|f(\cdot-h)-f(\cdot)\|^2_{L^\infty(\mathbb{T})} \leq \|f\|^2_{C^{0,\alpha}(\mathbb{T})}|h|^{2\alpha} \\
&= \|f\|^2_{C^{0,\alpha}(\mathbb{T})}\left(\frac{2\pi}{3}\frac{1}{2^m}\right)^{2\alpha}.
\end{aligned}$$

We now want to move from an $l^2(\mathbb{Z})$ to an $l^1(\mathbb{Z})$ sum to be able to control the absolute convergence. And for that we use the Cauchy-Schwarz inequality, noting that the sum has $2 \times (2^{m+1} - 2^m) = 2^{m+1}$ terms. Therefore

$$\sum_{2^m\leq|n|<2^{m+1}} |\hat{f}(n)| \leq 2^{\frac{m+1}{2}} \left(\sum_{2^m\leq|n|<2^{m+1}} |\hat{f}(n)|^2\right)^{\frac{1}{2}} \leq \|f\|_{C^{0,\alpha}(\mathbb{T})}\left(\frac{2\pi}{3}\right)^\alpha 2^{\frac{m+1}{2}-m\alpha},$$

[1] Observe again, as was also mentioned at the end of Lesson 17, that by grouping Fourier series in dyadic blocks of frequencies one obtains stronger results, of which the Littlewood-Paley theory is the paradigmatic example.

so that, finally

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)| = |\hat{f}(0)| + \sum_{m=0}^{\infty} \left(\sum_{2^m \leq |n| < 2^{m+1}} |\hat{f}(n)| \right) \leq |\hat{f}(0)| + \|f\|_{C^{0,\alpha}(\mathbb{T})} \left(\frac{2\pi}{3}\right)^{\alpha} \sum_{m=0}^{\infty} 2^{\frac{m+1}{2} - m\alpha},$$

which is finite for $\alpha > \frac{1}{2}$. Just note also that $|\hat{f}(0)| \leq \|\hat{f}\|_{l^\infty(\mathbb{Z})} \leq \|f\|_{L^1(\mathbb{T})} \leq \|f\|_{L^\infty(\mathbb{T})} \leq \|f\|_{C^{0,\alpha}(\mathbb{T})}$ to obtain the bound (1.1). □

The proof above cannot be improved for there exist examples of Hölder-$\frac{1}{2}$ continuous functions whose Fourier series do not converge absolutely. An example of which is the Hardy-Littlewood series (see Zygmund's book [3], pg. 197)

$$\sum_{n=1}^{\infty} \frac{e^{in\log n}}{n} e^{int}.$$

One can, nevertheless, relax the Hölder continuity a bit, compensating with bounded variation, obtaining a result due to Zygmund.

Theorem 1.4. (Zygmund) *Let $f \in C^{0,\alpha}(\mathbb{T}) \cap BV(\mathbb{T})$ for some $\alpha > 0$. Then $f \in \mathbb{A}(\mathbb{T})$.*

Proof. See Katznelson's book, pg. 33 in [1] or pg. 35 in [2]. □

Even though, from Bernstein's theorem and the Hardy-Littlewood counter-example, Hölder-$(\frac{1}{2} + \varepsilon)$ is the least regularity for which the Fourier series will converge absolutely, we will see soon that Hölder-α, for any $\alpha > 0$ is still enough to guarantee uniform convergence, even though possibly not absolute, for $0 < \alpha \leq \frac{1}{2}$.

Let us pause for a moment and compare these results to the convergence of Taylor series. In spite of the subtlety and apparent instability of Fourier series with respect to convergence, the fact is that they are absolutely convergent down to extremely low regularities of functions that are only continuous, and even nowhere differentiable. While Taylor series only converge absolutely in the interior of their radius of convergence, for functions that not only have to be infinitely differentiable, they have to be analytic. And functions with absolutely convergent Fourier series are, among the whole $L^1(\mathbb{T})$ space of functions under consideration, only still a very small and restrictive subclass.

References

[1] Yitzhak Katznelson *An Introduction to Harmonic Analysis,* 2nd Edition, Dover Publications, 1976.
[2] Yitzhak Katznelson *An Introduction to Harmonic Analysis,* 3rd Edition, Cambridge University Press, 2004.
[3] Antoni Zygmund *Trigonometric Series,* 3rd Edition, Cambridge University Press, 2003.